Learning Arithmetic by the Montessori Method

This booklet ist an attempt to present the elementary part of the Method in such way that it can be applied even by those who have not had Montessori Training. (Drummond, Margaret)

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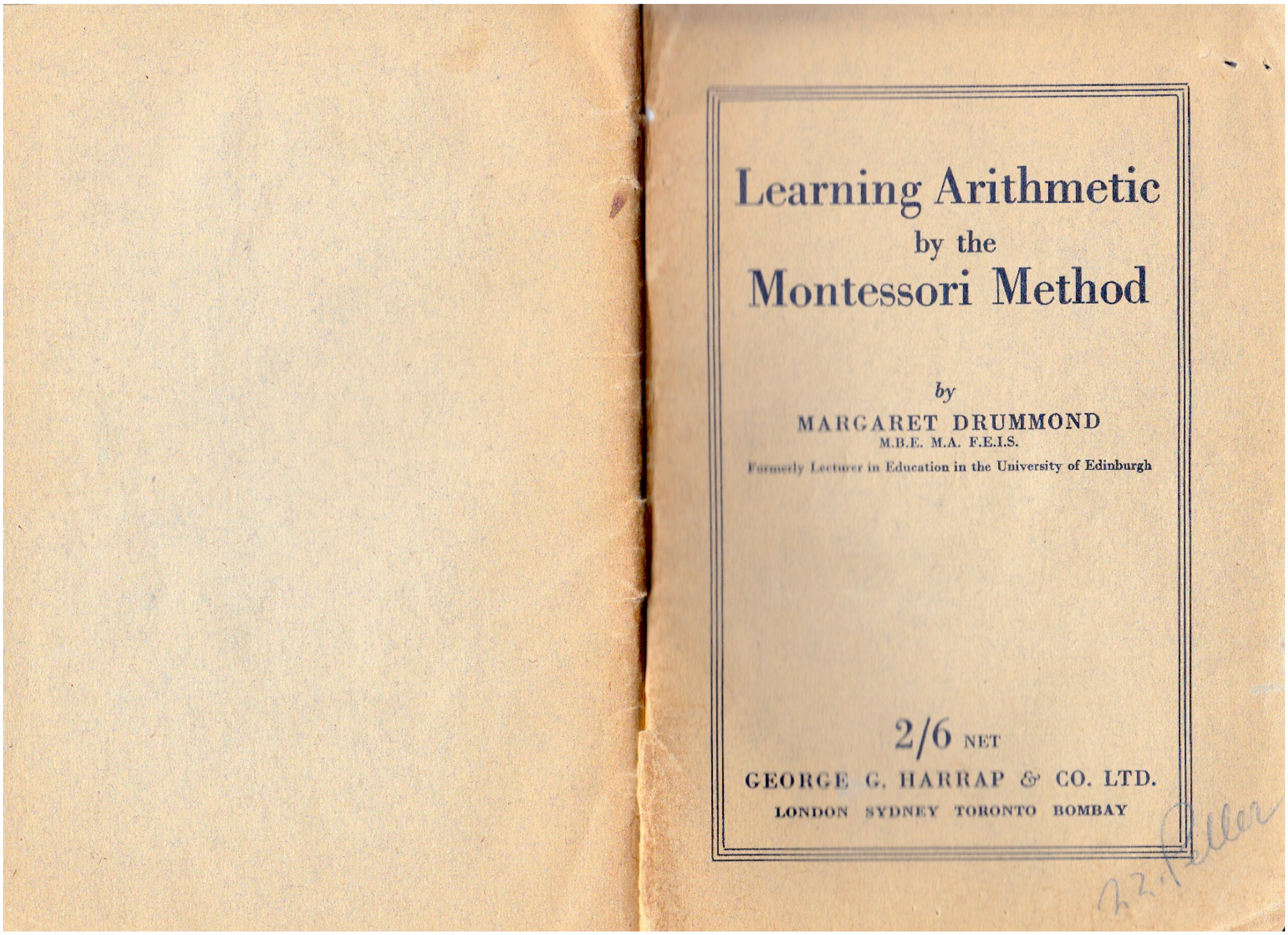

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by the
Montessori Method

by
MARGARET DRUMMOND
M.B.E. M.A. F.E.I.S.
Formerly Lecturer in Education in the University of Edinburgh

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LEARNING ARITHMETIC BY
THE MONTESSORI METHOD

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LEARNING ARITHMETIC
BY THE MONTESSORI METHOD

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MARGARET DRUMMOND
M.B.E. M.A. F.E.I.S.

FORMERLY LECTURER IN EDUCATION IN THE UNIVERSITY OF EDINBURGH

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THE MONTESSORI METHOD

In the teaching of arithmetic to young children account has to be taken of two facts: (1) ability to understand number and number relations depends more on mental development than on teaching; and (2) children acquire this ability at different ages, and need varying amounts of practice in order to develop it.

It may be added that insistent teaching before a child is ripe to receive instruction is certain to do harm. Many of the children coming to school for the first time at five years of age are not sufficiently mature mentally to benefit from any systematic instruction in number.

In the home and in the nursery school number ideas should be introduced, and number names should be used naturally and as occasions arise. The child himself will introduce the words into his conversation, and the beginning of understanding may be noted. In a favourable environment, where a child is continually adding to his knowledge and his vocabulary in connexion with his natural activities, he very commonly begins to show a sustained interest in number when he is about four years of age.

Nevertheless, in this subject individual differences, whether they are due to differences in early environment or to differences in natural ability, are so great that number work should never be pressed upon young children. If one goes into a classroom and finds all the six-year-olds doing or attempting to do the same exercise in number, one may be sure that many are doing work which is injurious or, at best, unprofitable to themselves.

Clinical experience with school children alleged to be 'arithmetical duffers' or 'without number sense' has convinced me that their mental confusion is due to the fact that they have not worked sufficiently with suitable concrete material, and so have no real intellectual appreciation of the basic facts on which our number system rests. Recently three children were referred to my clinic for weakness in arithmetic, particularly subtraction. Their average age was nine years, eight months. They were all intelligent children, two of them having I.Q.'s of not less than 112. Here are some specimens of their subtraction sums:

(*a*)	(*b*)	(*c*)	(*d*)	(*e*)
8354	250	46	54	6067
5676	49	7	39	5970
3777	211	40	20	0007

These sums are from a graded subtraction test published by Dr Schonell. This test contains fifty-six sums; of these the dullest of my three children (I.Q. 98) had the most correct, probably just because her subtraction habits were better; there is no reason to suppose that her understanding of what she was doing was at all superior. Sums (*a*) and (*b*) are clearly worked from the left instead of from the right; the 'borrowing' and 'paying back' are, therefore, quite irrational. Sums (*c*) and (*d*) seem to be done on the principle that you cannot take a number from one smaller than itself—therefore you set down 0 and proceed to the next column. The mistake in sum (*e*) is probably a slip, as other sums like this are worked correctly. The unnecessary 0's are set down on the instructions of the teacher, who considers that it 'keeps the children right.' All the children show a complete lack of understanding of the decimal system embodied in the columns.

The Montessori Method offers a system of instruction in arithmetic which fits the child. Its main principles are: (*a*) No formal teaching until the child's mind is seen to be ripe for it; it follows from this that instruction is, as a rule, given to one child at a time or to small groups. (*b*) The very early clear presentation of the decimal principle on which our number system rests. (*c*) The free use of concrete material until the child himself elects to abandon it. (*d*) Individual progress limited only by individual capacity. (*e*) Learning by experiment and thought rather than from the words of the teacher.

When we consider the present practice with regard to the teaching of arithmetic in most of our schools we see that these principles are revolutionary. They demand the abolition of class teaching, the abandonment of a fixed time-table, the introduction of very large numbers even to the youngest children, the provision in all classes of a plentiful supply of suitable concrete material, and the absolute rejection of the belief that all children in the same class should do the same sums at the same time or reach the same set goal at the end of the session.

This booklet is an attempt to present the elementary part of the Method in such a way that it can be applied even by those who have not had a Montessori training. Such a training is, however, of the utmost value, for Dr Montessori never concerns herself with only part of the child, but with his whole personality; and those teachers who make a beginning by following her Method in arithmetic should take every opportunity of seeing how the Method works when applied in entirety.

THE PRELIMINARY STAGE

The first material presented is a series of brightly painted wooden rods with a cross-section of 2 centimetres squared. The longest of these is 1 metre in length, and they decrease regularly by 1 decimetre. The decimetres are painted alternately red and blue (Fig. 1). The child is shown how to arrange these rods in order of length, and is taught how to count the sections. He has often heard the number names used in the ordinary routine of home and school, and the ideas he connects with those names now begin to gain precision.

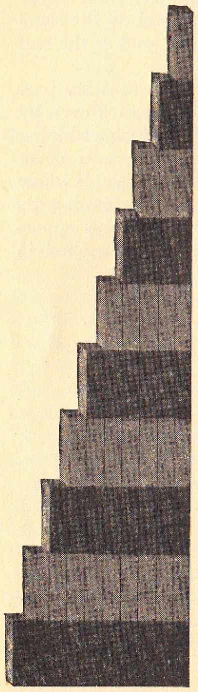


Fig. 1. Divided Rods

In arranging the rods the child comes to know not only the numbers but their relationship to one another: *five*, for example, is the fifth rod; *seven*, the seventh. He performs experiments: he finds that *seven* placed end to end with *two* is equal in length to *nine*; when he takes *two* from this newly formed *nine*, *seven* is left; if he takes away *seven*, *two* is left. He is teaching himself addition and subtraction.

He may be shown how to arrange the rods so as to get all the combinations making up ten, thus:

$9+1=10$
$8+2=10$
$7+3=10$
$6+4=10$

The *five* is left without a partner, but it is easy to see that by using it twice we should again obtain 10.

The child makes these discoveries. The adult can go farther; he sees that the sum of the series 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, is equal to $10+(9+1)+(8+2)+(7+3)+(6+4)+5$, which is equal to $10\times5+5=55$. This leads to the general formula for the sum of any such series:

$$n+\left(n-1\right)+\left(n-2\right)\ldots+1=\left(n\times\frac{n}{2}\right)+\frac{n}{2}=\frac{n^2}{2}+\frac{n}{2}=\frac{n(n+1)}{2}.$$

Generally speaking, a child who is at this stage has already learned to write by means of the sandpaper letters; he is now

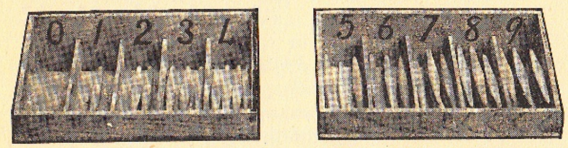


Fig. 2. Number Boxes

given sandpaper figures, is taught how to trace them with his finger, and to associate each symbol with the appropriate rod. The child thus comes to be able to write the figures. He also uses cards with the figures on them, and he should be taught the signs +, −, and =. He can set down his findings, thus:

[2] [+] [5] [=] [7] or [7] [−] [5] [=] [2]

All the work with the rods is done on a self-coloured mat or rug spread on the floor.

The next piece of apparatus stresses number as made up of loose units. It consists of two boxes (Fig. 2) with compartments labelled 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, along with 45 wooden spindles. The child is taught to place in each compartment the number of spindles indicated by the figure, and each bundle may be fastened together by a coloured ribbon or an elastic band. In doing this exercise he realizes that, while there are ten figures and ten compartments, there are only nine bundles of spindles, and he begins to understand the significance of zero.

As a third exercise we give ten little cards bearing the figures 0 to 9, and 45 small discs. The child places the cards in order, and under each he arranges the appropriate number of counters in double file, so that the interesting difference between odd and even numbers becomes apparent to him (Fig. 3).

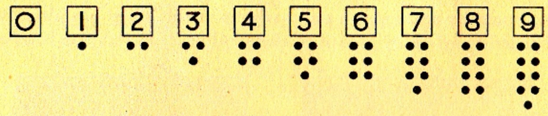


Fig. 3

When a child is perfectly secure in the performance of these three sets of exercises, (*a*) with the rods, (*b*) with the spindles, (*c*) with the counters, he has surmounted the first and most important step, and is ready to go forward confidently.

No regular instruction in arithmetic should be given to any child until he has passed this preliminary stage. The period during which these simple exercises interest the child is one of development rather than of mere learning, and the most we can wisely do, besides giving the opportunity and the material for these specific exercises, is to use number words in connexion with the events of life and to encourage the learning of the number names in order. Children delight in number rhymes and even in repeating the number sequence before they have more than the haziest notion of the meaning of the terms.

THE ELEMENTARY STAGE

The children now know quite well the significance of the small numbers and their relationship to one another; they can also recognize and write the corresponding symbols. They do not yet know that they have already learned all the symbols necessary to represent any number whatever, no matter of what magnitude, nor do they know that the symbol 10 stands for a *unit* which may be dealt with numerically just in the same way as the original unit. They know what 0 means when it stands alone, but they do not know what it means when it is used as a sign of what we call place value.

The simplicity and clarity of arithmetic are due to the brilliant idea of allowing position to give value to the number symbols. Thus the same symbol 2 may, according to its position, mean two units, two tens, two hundreds, etc. The base of our system being ten, the power to reckon enormous quantities is within the power of anyone able to count up to nine.

In the ordinary schools many children find insuperable difficulties in arithmetic because their minds have never arrived at a clear understanding of the basis on which it rests. Little children can gain this understanding only through much practice with the concrete, and the amount of such practice is often arbitrarily limited by teacher or inspector or by lack of suitable material.

In the Montessori school as soon as the child is quite at home with the preliminary exercises described above he is given a demonstration of the specially prepared material which shows him how any quantity can be reckoned without counting beyond nine. This material consists of single beads, ten-bead bars, hundred-bead squares, and thousand-bead cubes. All the beads are the same colour, a beautiful, translucent amber.

Along with these we provide number cards: green for the units; blue, twice as long, for the tens; red, three times as long, for the hundreds; and, for the thousands, cards four times as long, and coloured green like the units. If preferred, the cards may be white, and the figures coloured green, blue, and red. It will be seen that these cards can be superposed so as

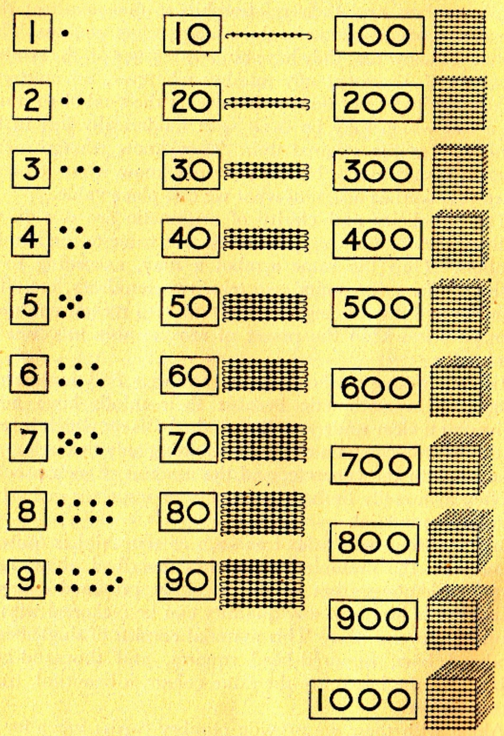


Fig. 4

to represent any number up to four figures. The cards for the units are identical with those used by the children when working with the wooden rods.

For the demonstration the material is laid out as shown in Fig. 4.

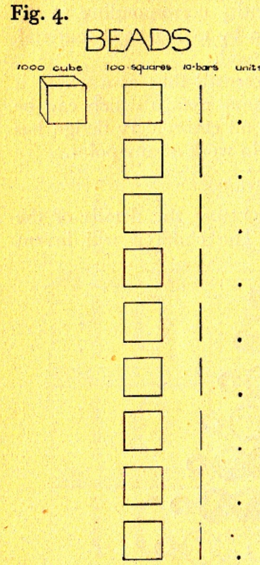


Fig. 5

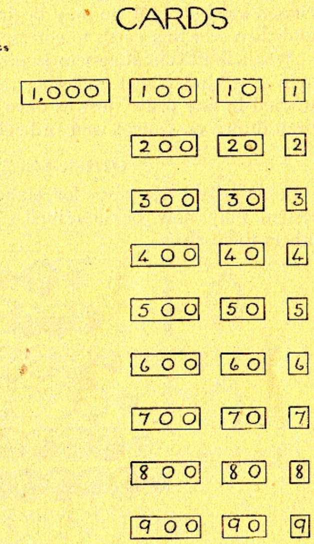


Fig. 6

The children delight in laying out this material, and in arranging it in various systematic ways. They can also use it for the composition of large numbers.

To make quite clear the fundamental idea of the decimal system, namely, that to be able to represent any number whatever we require only nine units of each order, we set out the material as shown in Fig. 5.

We also lay out the cards in order (Fig. 6).

Now, with either the cards or the material we can build any number we like from 1 to 1999. A child may take the cards 700, 50, and 3. Superposing them, he obtains the number 753. From the bead material he takes seven squares, five bars, and three single beads, the quantity corresponding to the chosen symbols. Or he may begin by taking bead material, and then selecting cards to correspond.

When the decimal system is presented thus as a whole it acts as a kind of net enclosing many details which can be studied in any order, just as once an embroidery design has been drawn on canvas we can begin work at any point.

OTHER EXERCISES

Some of the exercises for elaborating the details of the system may now be described. The children will invent others themselves.

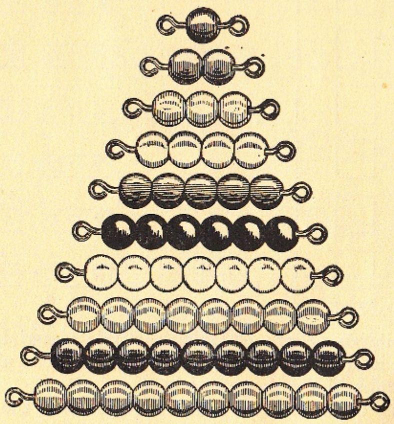


FIG. 7

To illustrate the all-important step through the ten we provide bead-bars of distinctive colours (Fig. 7); with these

the children are taught to build the series 1 to 10, and thereafter 10 + 1, 10 + 2, etc., until they reach two tens, two tens plus one, and so on.

The words *eleven*, *twelve*, etc., may be taught at this time. Many children will have learned them already in the ordinary course of life. We may, however, arouse special interest in

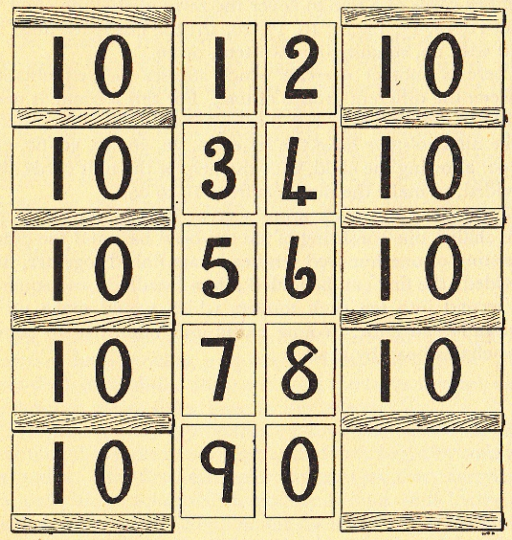


FIG. 8

these words and their meanings by providing little cards printed in two colours which the child may be taught to arrange in this way:

black	*red*	*black*	*red*	*black*	*red*
e	leven	four	teen	seven	teen
twe	lve	fif	teen	eigh	teen
thir	teen	six	teen	nine	teen

A*

Obviously the exercise might also be done by means of loose letters of two colours. Interesting questions about the etymology of the words might arise.

An exercise performed with the figures alone is now used. The material is shown in Fig. 8. It consists of a frame in which there are nine 10's with a vacant space at the bottom. Number cards of a size to cover the zeros enable the child to build 11, 12, 13, 14, 15, 16, 17, 18, 19; the last space must be filled with 20, standing for two tens, or 20.

There is another exercise which stresses the resemblance between the different groups of tens. For this a frame is used similar to that shown in Fig. 8 except that in this case, instead of the nine 10's, we have the series 10, 20, 30, 40, 50, 60, 70, 80, 90, allowing the child, with the help of the unit cards also provided, to make the numbers from 10 to 99.

Linear counting often arouses great interest at this stage. Two chains, one consisting of ten ten-bead bars and the other consisting of one hundred ten-bead bars linked together, are provided; the first can be folded into a hundred-bead square, the second into ten such squares which would make up a thousand-bead cube. Young enthusiasts often like to count these chains bead by bead.

THE FOUR FUNDAMENTAL PROCESSES

ADDITION. With the long rods the child has already performed operations of addition and subtraction. He has now to learn to deal with larger quantities.

The material required consists of a few light trays lined with green baize and a large quantity of cubes, squares, bars, and single beads, along with sets of number cards, one large, and others small. Cards from a small set indicating a number are laid on two or three trays. Each tray is given to a child, who goes to the bead store and selects the number of thousands, hundreds, tens, and units indicated by his cards. These are verified by the teacher, after which all the material brought is collected on the table. The teacher places the small cards in column—*e.g.*:

1342
2526
1121

The children now count the bead material and discover that they have 4 cubes, 9 squares, 8 tens, and 9 units. They have done an addition sum, and with the large number cards the teacher places the answer at the bottom.

The children have probably counted the thousands first; next time the teacher may suggest that they begin by counting the units. The first examples do not involve 'carrying'; gradually the teacher will introduce it, and with 'carrying' the fun really begins. Suppose the three trays bear the figures 2631, 1328, and 1536. When the bead material has been brought and counted the children find they have 15 single beads, 8 bars, 14 squares, and 4 cubes. They know, however, that they must not have more than 9 of any one order. They change ten of the single beads for a bar, and ten of the squares for a cube, and so find the answer is 5 cubes, 4 squares, 9 tens, and 5 units, or 5495.

The Snake Game. This is a favourite exercise which leads to familiarity with the addition combinations, particularly those which go to make up ten. The material required consists of a number of ten-bead bars, a number of bead-stairs, and a box containing ten black-and-white bead-bars.

On a self-coloured felt mat we place some of the coloured bead-bars, arranging them end to end to form a 'serpent,' say $5+6$. For the first ten beads we substitute a ten-bar; for the one bead not included we substitute one black bead, that is, $5+6=10+1=11$. Let us take a longer snake, say $5+6+8+3+4$. We substitute a ten-bar and the single black bead for the first two coloured bars. We then substitute a ten-bar and the black two-bead bar for the one, the eight, and the three. For the black two and the coloured four we substitute a six from the black-and-white stair. We now have two tens and a six—a golden serpent with a black-and-white head instead of the variegated one with which we started. The fact that $5+6+8+3+4=26$ can be verified by the actual counting of the variegated bars.

The children like this game very much, and often make serpents containing several hundred beads. Through such spontaneous work they acquire a remarkable mastery of number relations.

Addition Chart. Another attractive piece of apparatus, the aim of which is to familiarize the child with the addition tables, is a card ruled as in Fig. 9.

With this are provided blue and red strips of cardboard of lengths varying from one to nine, and a little basket containing slips of paper on which are the addition combinations from $1+1$ to $17+1$, with spaces for the answers. The child takes one of these slips, and by placing the appropriate strips on the chart he discovers the answer by looking at the figure in the top line above the end square of the second strip.

Many children between five and six years of age like, by means of this chart, to write out one by one all possible combinations.

The Dot Game. An attractive method of adding large numbers without the use of bead material is found in what has come to be known as 'the dot game.' We provide printed

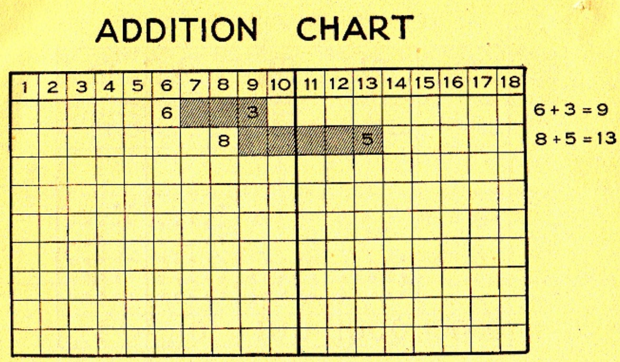


Fig. 9. Addition Chart

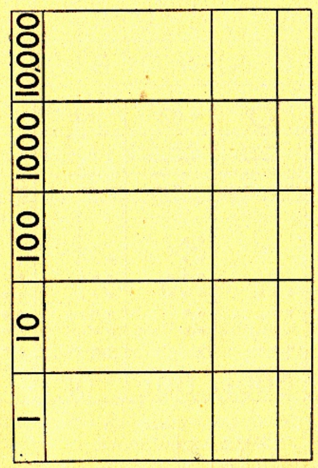


Fig. 10 (*see p.* 23)

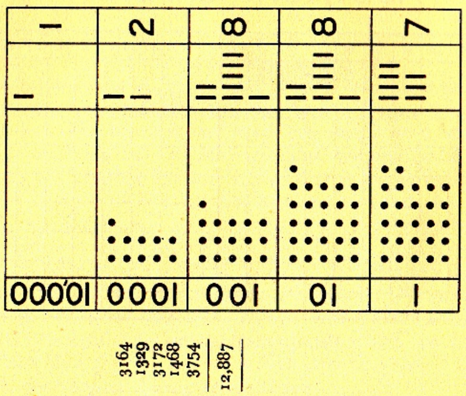


Fig. 12

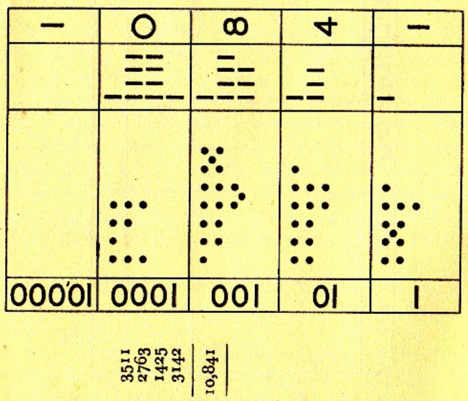


Fig. 11

forms in which compartments are assigned to the hierarchies (Fig. 10). The child sets down the sum in figures, and then proceeds to translate the figures systematically into dots, being careful to put the dots in the appropriate compartment. For

0	1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9	10
2	3	4	5	6	7	8	9	10	11
3	4	5	6	7	8	9	10	11	12
4	5	6	7	8	9	10	11	12	13
5	6	7	8	9	10	11	12	13	14
6	7	8	9	10	11	12	13	14	15
7	8	9	10	11	12	13	14	15	16
8	9	10	11	12	13	14	15	16	17
9	10	11	12	13	14	15	16	17	18

Fig. 13 (*see p.* 24)

every ten dots he has in any compartment he now puts a stroke in the first column to the right, in the higher compartment; dots fewer in number than ten are recorded by means of strokes in the right-hand column on the same level. If, now, more than nine strokes appear in any compartment they would have to be represented by 1 in the superior denomination when we translate our answer into figures. Calculation is facilitated if we arrange the dots in fives or tens. We give two examples; the first (Fig. 11) shows the numbers represented by separate groups of dots; in the second (Fig. 12) the dots are arranged vertically in lines of five.

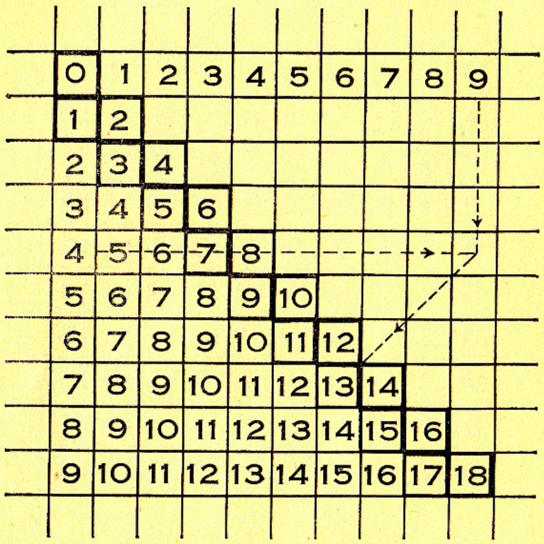


Fig. 14

Number Charts. The children are also introduced to the following number charts, which they study with great interest. The first (Fig. 13) shows all the possible addition combinations. In adding two numbers—*e.g.*, 3 and 4—the child moves his right forefinger down from the 3 in the top line, and his left forefinger along from the 4 in the first vertical column. They meet at 7, which is the required answer.

In Fig. 13 there are repetitions, for additions can be read either way: $3+7$ is the same as $7+3$. Contrary to adult expectation, this fact is not immediately obvious to the young child. He may convince himself of it by the use of Fig. 14, in

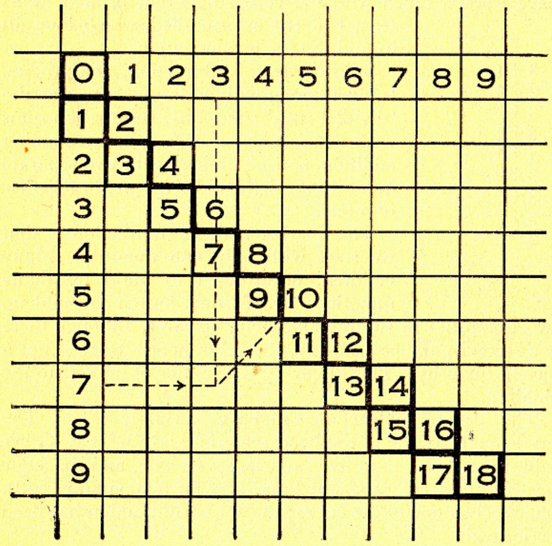


Fig. 15

which all the combinations occur, but none twice. If his fingers meet on a blank square he can get the required answer by reading the sum backward, $4+9$ instead of $9+4$. He can also obtain it on the diagonal drawn from the blank meeting-place. Hence we see that we can reduce the number of figures still further, as in Fig. 15, and yet have in the chart the solution of every two-figure addition combination. In this chart the fingers will usually meet on a blank square, and the answer will always be found on the diagonal—*e.g.*, $3 + 7 = 10$. Numbers have a magic of their own which always makes a strong appeal to the child.

When the children are familiar with the combinations and have worked much with the varied material, printed forms (*e.g.*, Fig. 16) are supplied, which they fill in and use for memorization.

$3 + 1 =$
$3 + 2 =$
$3 + 3 =$
$3 + 4 =$
$3 + 5 =$
$3 + 6 =$
$3 + 7 =$
$3 + 8 =$
$3 + 9 =$

Fig. 16

Subtraction. Subtraction, as a process, is taken along with addition. Even at the 'divided rods' stage little subtraction sums were done. Then it was seen as undoing addition, and obtaining again two separate quantities which had been brought together into one.

The characteristic of subtraction is that we start with only one quantity. Many children, accustomed to build up many quantities for addition, hasten to build up two quantities in subtraction. Here, however, although there is a second number—namely, the subtrahend—there is not a second quantity. This discovery is a matter of interest to the children.

The idea of subtraction can be very simply given. The teacher takes some of the bead material and places it on her table; say, two cubes, five squares, seven bars, and six single beads. Edith comes forward and asks for a thousand beads. She receives one of the cubes; the act of subtraction has been performed.

Take a more complex example: $1584 - 657$. A child places on the table one cube, five squares, eight bars, and four single beads. From this there are to be taken away six hundred and fifty-seven. In order to take away seven single beads we must exchange one of the ten-bars for ten single beads. We then have fourteen single beads altogether. When we take away seven, seven remain. From the seven bars which we now have, we take away five, leaving two. From the five squares we cannot take six. We must exchange the cube for ten squares, giving us fifteen squares in all. If we then take away six, we

shall have nine left. The result of the operation $1584 - 657$ is, therefore, 927—that is, the nine squares, the two bars, and the seven single beads that remain on the table.

The children realize that the original quantity has now been divided into two unequal parts, which, when brought together again, once more make up the whole.

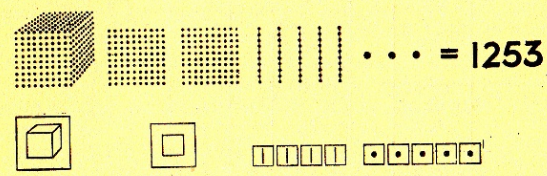


Fig. 17

To facilitate these early exercises in subtraction little coloured cards have been prepared which represent the quantities found in the subtrahend. They are like tickets which give a claim upon the material.

A subtraction sum may then appear as on the table in Fig. 17. This stands for $1253 - 1145$, the minuend being bead material, and the subtrahend being indicated by means of the cards.

Continuous subtraction can, of course, be done until the original quantity is exhausted—*e.g.*, 25 may be subtracted from 125 five times. Such continuous subtraction may be looked on as a preparation for division, just as continuous addition of the same quantity is a preparation for multiplication.

Other material used for subtraction consists of strips of paper on which units, tens, hundreds, etc., are printed in different colours (Fig. 18). Suppose we have the sum $7492 - 3241$; we tear off a sufficiency of the strips, and then with a pair of scissors cut off the subtrahend. This device shows the children very clearly that in subtraction we are dealing with only one quantity which we have to divide into two parts, one of which we know, the subtrahend, the other of which we have to discover. Obviously when these two are put together again, we have the original quantity; this is the proof of the correctness of our work.

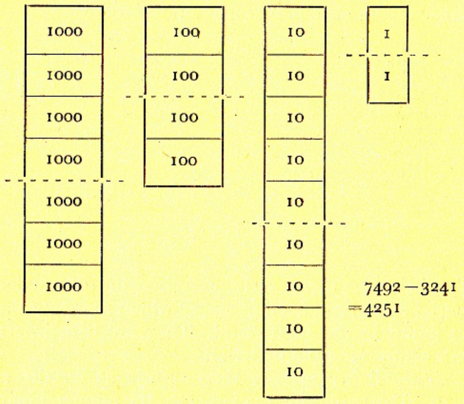


Fig. 18

When 'borrowing' is necessary, as, for example, in 5732 − 2815, we can exchange one of a higher denomination for ten of a lower, just as we did when working with the beads.

Multiplication. The difference between addition and multiplication is that in the latter we deal only with equal quantities. To introduce this idea to the children we ask three of them to bring to the table equal quantities. Suppose they each decide to bring one cube, four squares, seven bars, and three single beads. This material is then classified, and the necessary exchanges are effected. The final result is four cubes, four squares, one bar, and nine single beads, or four thousand

four hundred and nineteen. The children have performed a multiplication sum—namely, $1473 \times 3 = 4419$.

The children are very fond of using the coloured bead-stairs (Fig. 7), and in working with them they make many interesting discoveries. They find, for example, that two two-bead bars, three three-bead bars, four four-bead bars, etc., will all give squares in the same way as ten ten-bead bars. Also, three seven-bead bars contain the same number of beads as seven three-bead bars; that is, seven times three is the same as three times seven. This truth can be generalized, thus halving the labour involved in learning the multiplication tables.

One of the discoveries made by Dr Montessori is that LITTLE children like to do BIG sums, and, thanks to her early demonstration of the decimal system, there is practically no limit to the amounts with which they can deal intelligently. They set themselves sums so formidable that no teacher or text-book nowadays would venture to give them.

In this country there is a prejudice against allowing learners to use concrete material freely and for as long as they like. Dr Montessori, however, has found that children taught by her method spontaneously discard the didactic material when their knowledge is secure. In one school conducted by Dr Montessori's co-worker, Signorina Maccheroni, one little boy worked so long with the material that his parents became uneasy and urged that pressure should be put upon him to abandon it. However, a little more patience was exercised and nothing was said to the child. One day he began to do long sums on the board, making no use of material. A few days afterwards, another child carried past him a tray bearing the material for long division. The boy who had been such an expert in its use now glanced at it carelessly. "Oh, that," he said. "I have forgotten how to use it."

When the children clearly understand multiplication they may learn to do it with pencil and paper only. As in the other processes, the big numbers which they like are quite within their capacity.

Take this example: sixty-four thousand nine hundred and thirty-five taken four times. If we set this sum down in figures it would appear thus:

Thousands		Units		
Tens	*Units*	*Hundreds*	*Tens*	*Units*
6	4	9	3	5
6	4	9	3	5
6	4	9	3	5
6	4	9	3	5

The difficulty of dealing with these numbers is evidently not at all affected by the hierarchy to which they belong. It is just as easy to deal with thousands as with units. But it would obviously simplify our work if we could remember the relevant products. These are:

$$5 \times 4 = 20$$
$$3 \times 4 = 12$$
$$9 \times 4 = 36$$
$$4 \times 4 = 16$$
$$6 \times 4 = 24$$

We know that we cannot have more than 9 in any denomination. Hence the 20 stands for no units and two tens, the 12 for two tens and one hundred, the 36 for six hundreds and three thousands, and so on. The result may be arranged thus:

Thousands			Units		
Hundreds	*Tens*	*Units*	*Hundreds*	*Tens*	*Units*
2	1 + 4	3 + 6	1 + 6	2 + 2	0

That is, 259,740, or two hundred and fifty-nine thousand seven hundred and forty.

To help the child to memorize the multiplication tables the following material is provided:

(*a*) A square board in which are one hundred depressions arranged in rows of ten, each one capable of holding a bead (Fig. 19); an attractive little box containing a hundred beads; a red counter; ten guide-labels with the numbers 1 to 10 printed on them in red.

With this material the child is shown how to build up his tables; take, for example, the five times table; the label 5 is

inserted at the left-hand side of the board, and the red counter is put above the figure 1 at the top; five beads are now placed in the depressions vertically below the figure 1; $5 \times 1 = 5$. The red counter is now moved to the second column, and five more beads are placed vertically below it; $5 \times 2 = 10$. This work is continued until the table is completed up to 5×10.

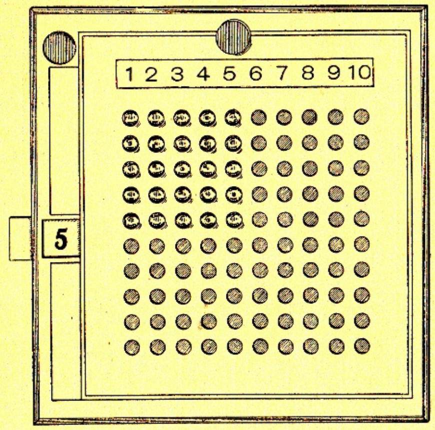


Fig. 19. Five times Five

(*b*) Printed forms ready for completion by the child similar to those used for addition (Fig. 16). As he works with the beads the child sets down his results on these forms. They are made up in packets of ten so that he may have plenty of practice.

(*c*) Certain numerical charts summarizing the work and helping the child to make many interesting discoveries. In the Table of Pythagoras (Fig. 20) he finds a summary of the multiplication tables as far as ten times ten. He can find the answer to any given combination—*e.g.*, 3×7—in the same way as he found the solution to the addition combinations

(Fig. 13). And because multiplication combinations are reversible, just as addition combinations are, approximately half of the table might be cut away. Some children will discover that the numbers along one of the diagonals are all

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

FIG. 20

squares, that the digits in the nine-times table always make nine when added together, and so on.

There is another chart (Fig. 21) which may be used for building up the tables by means of counters, and so discovering that each has its distinctive pattern. A comparison of the two charts will show that while every number from 1 to 100 occurs in the second, some numbers do not occur at all in the first.

Why is this, and what are these numbers? One great advantage of studying arithmetic in this free and individual way is that such questions can and do arise in the minds of the children.

1	2	3	4	5	6	**7**	8	9	10
11	12	13	**14**	15	16	17	18	19	20
21	22	23	24	25	26	27	**28**	29	30
31	32	33	34	**35**	36	37	38	39	40
41	**42**	43	44	45	46	47	48	**49**	50
51	52	53	54	55	**56**	57	58	59	60
61	62	**63**	64	65	66	67	68	69	**70**
71	72	73	74	75	76	**77**	78	79	80
81	82	83	**84**	85	86	87	88	89	90
91	92	93	94	95	96	97	**98**	99	100

FIG. 21
The seven-times table is shown.

This multiplication material is received with enthusiasm. Many combinations are retained in the memory through the intensive work with the boards, but, not content with this, children may often be seen walking up and down with their leaflets in their hands, going over and over the tables they have written out.

Division. In division a quantity is divided into equal parts which may be two or more than two in number; in subtraction the parts are two in number, and are usually unequal.

To bring the meaning of the term home to the children we place on the table two cubes, six squares, two bars, and six single beads. Two children are called, and, taking turns, they possess themselves, one at a time, of one cube, three squares, one bar, and three single beads.

Similarly, three children may be called to divide equally among themselves nine cubes, six squares, three bars, and four single beads. In this case, when each child has taken three cubes, two squares, one bar, and one single bead, there is one bead left, which cannot be divided. This is the remainder.

We now advance another step. We place on the table two cubes, eight squares, two bars, and four units. This is to be divided by four. Four children come forward. They quickly see that there is not a cube for each of them; hence the only thing that can be done is to exchange each cube for ten squares. There are now twenty-eight squares altogether; these the children proceed to divide, securing seven each. The two bars must now be exchanged for single beads, which brings the number of single beads to twenty-four. When these are equally divided each child finds he has six. That is, 2824 is made up of 4 times 706, or $2824 \div 4 = 706$.

The little ones greatly enjoy this activity. They love coming forward to take the objects one by one, and they love making the necessary exchanges—cubes into squares, squares into bars, and bars into single beads. Moreover, this exercise calls for a division of work as well as a division of a quantity. In practice a child is appointed who takes no part in the division, but simply looks after the exchanges, for which purpose he takes from the store a sufficient quantity of squares, bars, and single beads. He thus becomes the banker. It has been found desirable to appoint also a director, who looks after the distribution of the dividend, sees that each receives his fair share, and also checks the exchanges, and sees that no one interferes with the remainder. This social organization adds much to the children's pleasure in the activity.

The next advance comes when the divisor consists of more than one figure. Suppose we place on the table two cubes, eight squares, seven bars, and eight single beads, and call thirteen children to come and take part in the division. They crowd eagerly round the table, the result being confusion and disorder. We ask ten of them to stand aside and elect a representative, who will be given what is due to all ten. To this representative we give a big red bow, because he is a Ten; to the modest units we give little green girdles, to each of the nine excluded ones, who are standing aside, a little crestfallen, we give little white girdles.

The distribution now begins to the Red and the Greens. To the Red is given a cube to be divided among the ten children that he represents, each of whom therefore will receive one of the ten squares for which the cube will later be exchanged; the three Greens must now come forward and each take one square. The second cube is now given to Red, and again the three Greens each receive a square. Two squares, seven bars, and eight beads remain on the table. Red now takes a square which will provide a bar for each of his group, so the three Greens each take a bar. This can be repeated, leaving one bar with the eight single beads. Red now takes the bar, and the Greens each take a bead. Five beads are left.

Red now divides what he has equally among the ten children that he represents, each receiving two squares, two bars, and one single bead; the five remaining ones are left on the table. This division sum has been worked:

$$2878 \div 13 = 221 + R5.$$

The quotient, 221, is the amount received by each participant in the distribution.

All this time we have been concerned with pure arithmetic, and with the help of the material provided the children have studied it enthusiastically, and become familiar with its fundamental processes.

But all the time they are having experience of arithmetic—its terms and processes in relation to life. From their infancy they have known they have two hands, two eyes, one nose, ten fingers, and so on. They go shopping with their mothers, and become familiar with the coins of the realm. In school they are weighed and measured. There may be a savings bank in school to which they bring their contributions. Pencils are distributed (division), and collected again (addition) at the end of the morning.

The children are interested in this practical aspect of life, and it has been found that they will produce problems themselves in the form of stories or compositions. Here, for example, is the substance of a story written by a child in a Dutch school.

Some children had secretly eaten some of a number of beautiful apples that had been set aside in a basket. They then learned that these had been thirty apples set aside to be given to Mrs X on her thirtieth birthday. Well, the missing apples must be replaced. But how many have been eaten? Nobody can remember. Then a child exclaims, "Oh, it does not matter. We can find out by doing a subtraction sum."

Some knowledge of fractions is often found among children of nursery-school age. They are familiar with the expressions 'half of that,' or 'a quarter of this.' They 'go halves' with another child in the case of some piece of cake or other dainty. They hear of half a pound of tea, a quarter of a pound of butter, half a pint of milk, and so on.

To render exact the understanding of such terms and to bring the general idea of fractions into the intellectual life of the child Dr Montessori provides ten metal plates with mobile sections, as shown in the illustration (Fig. 22).

By experimenting with this material many discoveries can be made—*e.g.*, the more parts you have in a circle, the smaller is each part; a half is equal to two quarters, or three sixths, but to no exact number of fifths. A whole circle may be made up in a large variety of

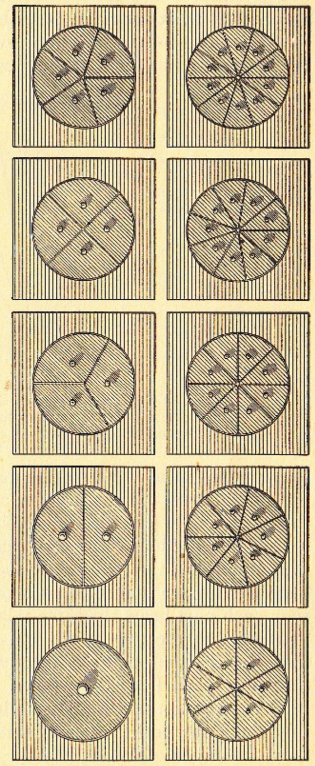


FIG. 22

ways; one half and two fifths do not quite make up a circle; you may discover one tenth will complete it, and also that one tenth is half a fifth.

The children are taught how the pieces are named, and also how to write them, and they begin to make sums, such as $\frac{3}{4}+\frac{1}{4}=1$, or $\frac{7}{8}-\frac{1}{2}=\frac{3}{8}$. The answers they find by studying the pieces, and to them they are real discoveries.

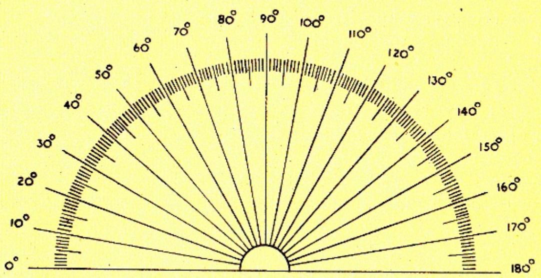


FIG. 23

The diameter of the circle is 10 cm., corresponding to metal insets. To facilitate placing of inset and reading of measurement the figures 0° to 180° are printed beyond the circumference of the circle.

By means of a white cardboard protractor (Fig. 23) they are taught how to measure the angles, and they learn that the whole circle is made up of 360°.

Decimal fractions come very easily to the children, because their numerical thoughts are so soaked in the decimal system. They readily see that if a unit is divided into ten equal parts, then each part bears the same relation to the unit as the unit itself does to the ten, or the ten to the hundred. They see that the decimal point is simply a way of separating the fractional parts from the wholes. Very small beads may be used to represent tenths. The children can imagine how very tiny the hundredths, the thousandths, etc., become, just as they can imagine how rapidly the sizes grow at the other end of the scale.

When there is this clear understanding of the decimal system as a whole the addition, subtraction, multiplication, and division of decimals present no difficulty at all.

To transform vulgar fractions into decimals we use a circle divided into a hundred equal parts (Fig. 24). If we place any

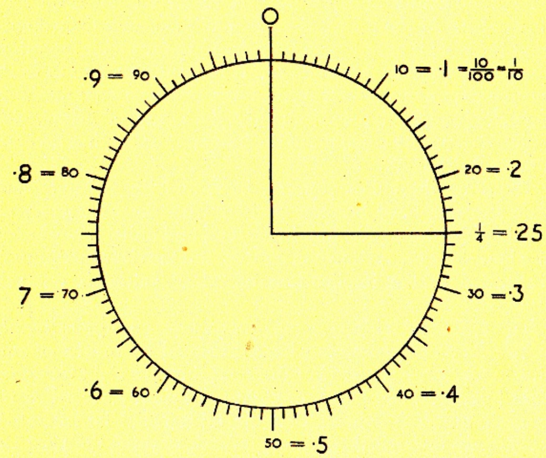


FIG. 24. INSTRUMENT FOR REDUCING VULGAR FRACTIONS TO DECIMALS
Diameter of circle is 10 cm. Circle is divided into a hundred equal parts.

section with its radius against the zero-line, it is easy to read off its value in the decimal form; in the illustration one quarter is shown to be equal to twenty-five hundredths, or ·25.

All this number work which has been described the children do spontaneously and joyfully. It is not set down on any time-table. At the time it is presented it becomes their dominant interest; if at any time they desire to turn their attention elsewhere, they are free to do so. It is natural to man to work in such dominant-interest spells, but our educational methods have forgotten this; consequently the study of arithmetic, which is naturally a delight to the human intellect, has become a painful drudgery.

Handling the cubes, squares, bars, and single beads, our little students have become familiar with the hierarchies of number, the thousands, hundreds, tens, and units. They know also how these are set down in figures. But the clear intellectual realization of the implications of 'place value' is not yet attained.

The following table (Fig. 25) shows how any quantity, no matter how enormous, can be represented by means of our nine numerical symbols. For instance, 8 means eight millions, eight hundreds, eight tens, or eight units simply according to its position. It is evident that the table could be extended indefinitely towards the left, or, indeed, towards the right, if we admit decimal fractions.

Now, children do not learn by means of explanations, although they may appear to listen to them with great attention. They require active work, by which the meaning gradually permeates their thought and becomes part of the very structure of their mind.

The required material is provided in the number frames (Fig. 26). In these colour becomes the symbol of rank. It will be seen that there are ten beads on each wire; the units are green, the tens blue, and the hundreds red. Along with the frames are given sheets of prepared paper (Fig. 27), on which the children write the figures 1, 2, 3, etc., as they shift the beads one by one from left to right. When the tenth bead

THOUSANDS OF MILLIONS			MILLIONS			THOUSANDS			SIMPLE UNITS		
HUNDREDS	TENS	UNITS	HUNDREDS	TENS	UNITS	HUNDREDS	TENS	UNITS	HUNDREDS	TENS	UNITS
5		2		3				1		2	

Fig. 25. How 502,030,001,020 is made up
The headings of the columns make it easy to translate the figure into words.

on the units line is pushed over all must be returned for one bead on the tens line, which is symbolized by 10. Then one unit bead is pushed to the right. The unit bead plus the tens bead stands for 11, and so on.

Very large numbers can obviously be set out on the frame, and then easily translated on to paper.

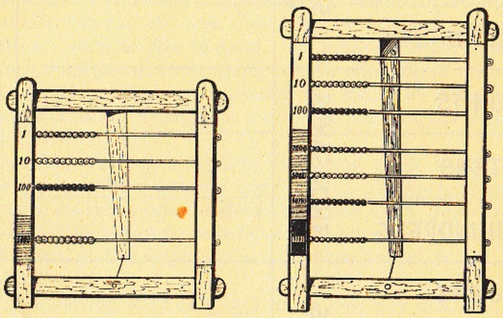
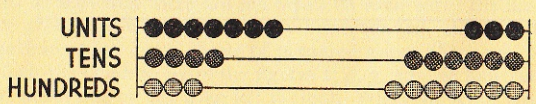


Fig. 26. Number Frames

The frame makes addition and subtraction very easy, provided we remember that whenever ten beads on any one line are pushed to the right we must return them and take instead one bead of the next hierarchy.

In subtraction 'borrowing' involves exchanging one bead of a higher denomination for ten of a lower. For this process most people prefer what is known as the method of equal additions, which can, in effect, be thus demonstrated on the frame. Take the example: 347 — 159.

We first arrange the beads thus:



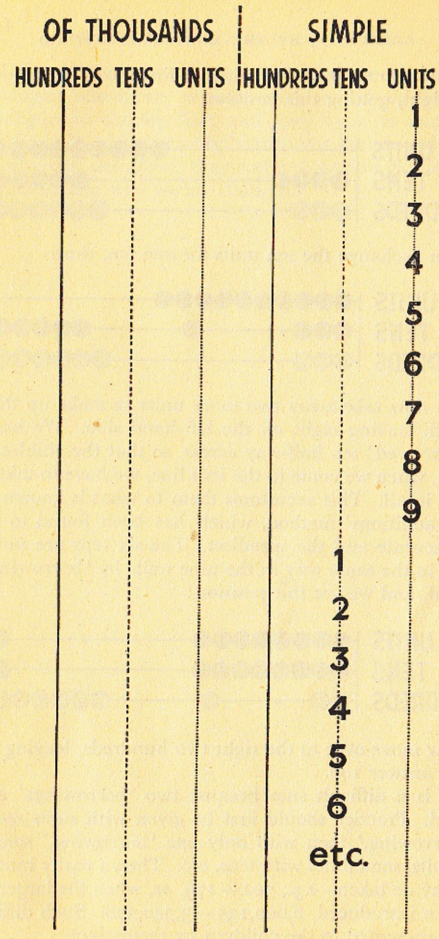
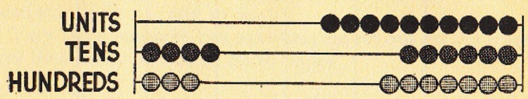


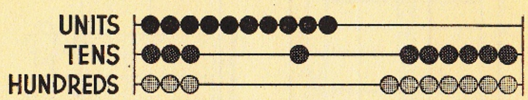
Fig. 27. Prepared Paper

Later the children will be taught to insert zeros. The vertical lines are coloured green, blue, and red.

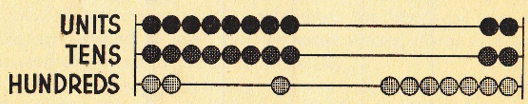
Then, to subtract the nine units, we first move the seven units to the right, getting this position:



We then exchange the ten units for one ten, thus:



We can now take away two more units to make up the nine required, leaving eight on the left-hand side. We have left the 'borrowed' ten half-way across, so that the children may see that, when we come to the tens line, we have to take away six tens in all. This accustoms them to what is known as the 'equal additions' method, which has been found to be the most accurate and the speediest. The six tens are now subtracted in the same way as the nine units by 'borrowing' one hundred, and we get this position:



We now move over to the right two hundreds, leaving on the left the answer 188.

This is a difficult sum because two 'borrowings' are demanded. Practice should first be given with sums requiring no 'borrowing,' then with only one 'borrowing,' sometimes with units, sometimes with tens, etc. Then a really interesting case may be taken—*e.g.*, 302 − 179, or, when the larger frame has been introduced, 6,000,135 − 3,542,378. Such difficulties will be presented to the children by themselves.

Intelligent children, learning subtraction in the way it is often taught, sometimes wonder why we should 'borrow' from a figure in the top line, and 'pay back' to one in the bottom line. Working peacefully with the Montessori frames in the way just described, they come to realize exactly what we are doing and why we are doing it.

LONG MULTIPLICATION

The children have built up, written, verified, and learned the multiplication tables. They can write the figures, and know the ingenious device of place value. Now they have to learn how to multiply by large numbers.

With the help of the frame it is easy for them to see that to multiply by 10, 100, 1000, etc., is simply a matter of changing the beads from one wire to another; 4×10 means changing 4 beads on the units wire for 4 on the tens wire; 40×100 means changing 4 beads on the tens wire for 4 on the thousands wire, giving $40 \times 100 = 4000$. If we multiply by 20, 200, 2000, etc., clearly we must get twice as many in the product. This may, of course, necessitate adjustments due to carrying—*e.g.*, $4 \times 500 = 4 \times 100 \times 5$. The four unit beads are exchanged for four on the hundreds wire, and then we have to take five times these four—*i.e.*, 20, or two tens. Now ten on any wire must be changed to one on the wire of the next higher denomination; therefore, in this case, we must put two beads on the thousands wire, and we find $4 \times 500 = 2000$.

Adults learning to use these frames sometimes find the work very difficult. The reason is that their multiplication has been established as a habit, and quite probably they have never fully understood the reason for what they do automatically. Such adults are often quite unable to believe in the value of the frames until they see children, prepared in the way we have described, working with them.

When these explanations have been given and assimilated by thought and practice the children are taught to write down the analysis of a multiplication sum in which the multiplier has more than one figure—*e.g.*, 348×23.

This means 348 taken 20 times + 3 times. The analysis is:

8 units		24 units	8 units		16 tens
4 tens	×3 =	12 tens	4 tens	×20 =	8 hundreds
3 hundreds		9 hundreds	3 hundreds		6 thousands

After the analysis is thus set out on paper the very pleasant and interesting work with the frames begins. In the first operation, 8×3, we have 24 units, that is, 4 on the units wire, and 2 on the tens wire; the 12 tens means 2 more on the tens wire, making 4 in all, and 1 on the hundreds wire. To this we have 9 hundreds to add, making 10 on the hundreds wire, which must be exchanged for 1 on the thousands wire. In the second group of operations we begin with 16 tens, or 6 tens and 1 hundred. The 6 tens added to the 4 already on the tens wire make 10 tens, which must be exchanged for 1 hundred. To the 2 hundred now on the hundreds wire we add the 8 hundred resulting from the multiplication, making 10 hundred, or 1 on the thousands wire, making with the 1 already there 2 thousand; to this falls to be added 6 thousand, so that our final answer is eight thousand and four, or 8004.

All this work is done very quickly on the frame—so quickly that a child working in this way can often obtain the answer to a long multiplication sum more quickly than an adult using the ordinary method. The difference between the two processes is that with the frame the addition of the partial products is made as each is obtained instead of being saved up for the end as is usual with us. After the child has had sufficient practice with the frame it is very easy to show him the usual way of setting down and working a multiplication sum.

For multiplication another frame is provided (Fig. 28) with a slightly different technique. In this case the multiplicand, say 4215, is written on a slip of paper which is laid on the lower margin of the frame *AB*, the figures corresponding with the appropriate wires. The figures of the multiplier, say 347, are separate, and are laid one by one, as we use them, on the right-hand margin of the frame; the units are placed at *C*, the tens at *D*, and the hundreds at *E*. First we let all the beads slip to the top of the frame, then we lay it flat on the table, and arrange our multiplicand and our first multiplier, 7, as directed. Now with the help of the beads we multiply by 7. At the end of this operation we have moved to the lower half of the frame beads representing 29,505. We have now to multiply by 40. We place the 4 representing 4 tens on the frame at *D*, and we move the multiplicand one space along, thus effecting multiplication by 10. We now multiply by 4, adding the products to the beads already forming part of the answer. We proceed in the same way with the 300, first placing the 3 at *E*, then moving the multiplicand another space to the left, then multiplying and adding as before. At the end the beads will be found to represent one million, four hundred and sixty-two thousand, six hundred and five, or 1,462,605.

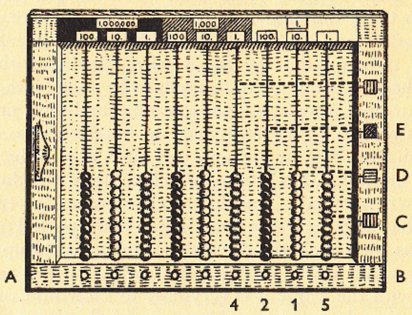


Fig. 28. Multiplication Frame

LONG DIVISION

At this stage the child uses for division boards similar to the multiplication boards (Fig. 19), beads, and blank forms (Fig. 29) on which he can record his results. Suppose he wishes to divide 32 by 7. On his board he places vertically under the figure 1 a row of seven beads; he does the same under figure 2, and so continues till his beads are exhausted. He finds he has four rows of 7, and four beads left over. This

Division	Remainder
÷ 2 =	
÷ 3 =	
÷ 4 =	
÷ 5 =	
÷ 6 =	
÷ 7 =	
÷ 8 =	
÷ 9 =	
÷ 10 =	

Fig. 29

result he records on his form. Attractive packets, each containing a hundred blank forms, are provided.

For long division the material is more complex, but extremely attractive; it interests the child deeply, and his diligent and spontaneous use of it results in a wonderful understanding and mastery of the process.

The material is shown very clearly in the illustration (Fig. 30). In each of the test-tubes are ten beads; these are of three different colours representing units, tens, hundreds. In the little boxes, which are placed in front of each test-tube rack, are placed beads to the amount of the dividend. If this were 8,798,589, then in the first box to the left we should put eight beads, taken from a test-tube in the rack behind, in the second box we should put seven beads from one of the second set of test-tubes, and so on till every box has received its quota. To divide this by a two-figure number, say 42, we require also two of the division boards. These we place at the left-hand side, and with their help begin our calculation.

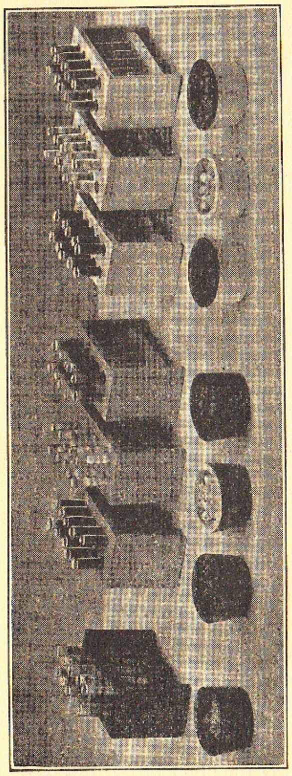


FIG. 30. LONG-DIVISION MATERIAL

In the slot of the first board we insert the figure 4, and in that of the second the figure 2. From the first box we take four beads and arrange them vertically on the first board as in short division; from the second box we take two beads, placing them similarly on the second board. There are now four beads left in the first box and five in the second, so we can repeat the operation just described, leaving no beads in the first box and three in the second. The first figure in the quotient is 2, which we write on our paper. The beads on the boards are restored to their test-tubes, the empty box is set aside, and the boards are pushed one step along. In the box which is now first we have fewer than four beads, so division by

four is not possible. We set down 0 as the second figure of the quotient, move the empty boards along another step, and exchange the three beads in the box for thirty from the next set of test-tubes, which beads we add to those in the box which was originally third and is now first; the second, now empty, is also pushed aside.

Our first box has now thirty-nine beads in it, and the one next to it has eight. We start our division again, getting four rows of four on the first board and four rows of two on the second. The second box is now empty, but we still have twenty-three beads in the first box. What must we do? Clearly we must change one of the twenty-three into ten of the lower denomination, and then continue the process of division. We get five more rows on each board when we find the second box again empty, and we have to supply it again with ten in the same way by drawing one from the first box and making the exchange.

Now, only one is left in the first box, so we can go no further until we exchange it also for ten of the next denomination. We put down 9 as the next figure in our quotient, clear the boards, and move them one step along to the left. In the box that is now first we have twenty beads, and in the one next to it five; with these we proceed as before, and so on until we come to the final remainder, nine beads in the last box.

These long-division exercises afford the children profound satisfaction, and by many of them are continued for a long time. Sooner or later, however, every child discards this concrete help, and begins to do long sums on the blackboard or with pencil and paper only.

PRIME NUMBERS, HIGHEST COMMON FACTOR, AND LEAST COMMON MULTIPLE

The child now performs certain exercises without the help of concrete material. One of the first of these is to continue the multiplication tables up to a point where the product is or is near one hundred. He is later given printed tables by means of which he may check his results.

As another exercise he receives long slips of paper on the left-hand side of which are printed numbers 1 to 100. Opposite to each he writes all the factors he can find in the tables, thus:

1
2
3
$4=2\times2$
5
$6=2\times3=3\times2$

and so on. From this table the prime numbers may be picked out.

Experiments or investigations may now be made with the beads. For example, six beads are taken. They may be arranged either in this way: *** ***; or in this: ** ** **. That is, we may consider 6 as two times 3, or as three times 2. That is, $3\times2=2\times3=6$. It is clear also that when we divide 6 beads into two equal groups we have 3 in each, and when we divide them into three equal groups we have 2 in each. As he meditates on such discoveries, the relationship between multiplication and division becomes clear in the child's mind.

Large numbers may be reduced to their prime factors, thus, $45=3\times15=3\times3\times5$. When results like this are obtained it is easy for the child to appreciate the convenience of the notation $2\times2=2^2$, $2\times2\times2=2^3$, etc.

About this time, too, an intensive study is made of all the bead material found on the special frame (Fig. 31). On the top shelf are the cubes of the first ten numbers; on the shelves

to the right are sufficient squares to build these cubes; in front of the squares lies a chain containing sufficient beads to build a square; and hanging on the little hooks under the top shelf are chains corresponding in number to the cubes. From handling this beautiful material and meditation thereon, the children gain a profound understanding of the relationships among numbers.

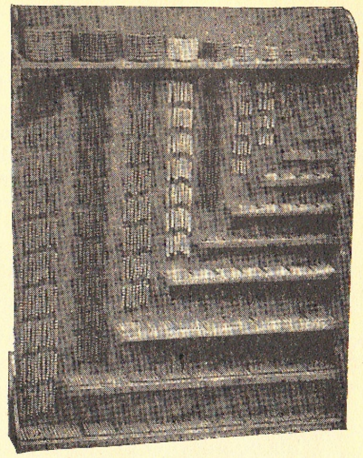
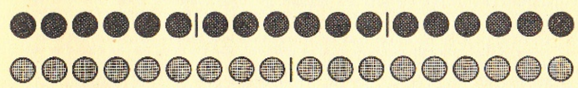


FIG. 31

When two or more numbers have been factorized it is easy for anyone to pick out the common factors, and so arrive at the H.C.F. and the L.C.M. It is also illuminating to study this matter with the aid of counters. For example, suppose we wish to find the L.C.M. of 6 and 9. We take two sets of counters, red and blue, and lay them out in groups of 6 and 9.



After we have laid three groups of 6 and two groups of 9 the two lines are the same length for the first time; therefore, the number of counters in this line, namely 18, is the least common multiple.

TWO OTHER METHODS OF MULTIPLICATION

Two other methods of multiplication have been found to interest the children and to increase their understanding.

For the first we require a board covered with squares of felt in two different colours, as in the illustration (Fig. 32).

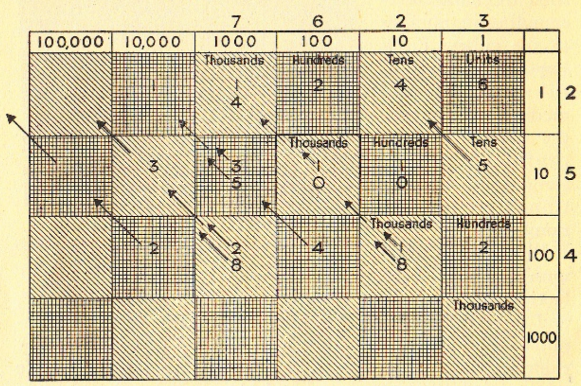


FIG. 32. MULTIPLICATION BOARD (7623 × 452)

For convenience figures are used here instead of the coloured bead-bars which the children use. When the sum has reached the stage indicated above the bars are pushed up diagonally to the top row. The number of beads in each square is as follows: 2 (outside the diagram), 8+2+3 (first square of diagram), 4+5+3+1, 8+1+1+4+1, 2+1+2, 5+4, 6. No square may have more than nine beads—*e.g.*, ten of the fifteen beads in the 1000 compartment must be removed, and for them one be placed in the adjacent compartment (10,000). Similar exchanges must be made in the case of the 10,000 square and 100,000 square. When the number of beads is less than ten we take a single bar to represent the sum. Finally our squares are like this: |3|4|4|5|5|9|6| —which is the product required.

The side of the squares should measure seven centimetres. We require also a considerable number of the coloured bead-stairs. At the top of the board and at the right-hand side are

numbers denoting the hierarchies; beyond these are placed the figures of the sum we wish to do—7623 × 452.

If we consider the squares, beginning at the top right-hand square, any bead placed in this will be a simple unit, and any bead in the square below will be a ten, as will be a bead in the square diagonally to the left of it. In short, each square indicates the value of a bead placed in it, and the squares lying diagonally from top left to right confer the same values.

We now begin our multiplication. Two times 3 is 6, that is, 6 units; we place a six-bar in the top right-hand square. Two times 2 is 4. This time the 4 means 4 tens; therefore a four-bar goes in the tens square. Two times 6 is 12; that is, a two-bar in the hundreds square, and a unit in the thousands square.

After we have finished the first line, we do the second line, multiplying by 5 in the same way, and then the third, where the first product is twelve hundred, or two hundreds and one thousand.

When we have finished the three lines we push the bars up diagonally, assembling them in the top line, and then proceed to make the exchanges required by the rule that we can never have more than 9 in any denomination.

The facility that results from working with this board is very remarkable. It is recorded of a boy about eight years of age that he one day brought to his teacher a sum something like this, saying, "I can do this all in one line."

$$\begin{array}{r} 5264 \\ 36 \\ \hline \\ \hline \end{array}$$

She asked for a demonstration. "Well, you see," he replied, "six fours are twenty-four, that is, four units, which I put down, for they are the only units, and two tens. Now you get tens by multiplying six by six and four by three, that is forty-eight, or, with the two we have already, fifty tens; so we put 0 in the tens place, and have five for the hundreds place.

1	2	3	4	5	6	7	8	9
10	20	30	40	50	60	70	80	90
100	200	300	400	500	600	700	800	900
1000	2000	3000	4000	5000	6000	7000	8000	9000
10000	20000	30000	40000	50000	60000	70000	80000	90000
100000	200000	300000	400000	500000	600000	700000	800000	900000
1000000	2000000	3000000	4000000	5000000	6000000	7000000	8000000	9000000

FIG. 33

Now you get hundreds by multiplying two by six and six by three; that is thirty, which gives us thirty-five hundreds altogether. . . ." And so he continued, until the whole answer was obtained.

This discovery of cross-multiplication, as it may be called, is a typical consequence of the intensive and spontaneous work of the children. When a process has been demonstrated and understood the teacher withdraws, leaving the child free to make his own deductions and discoveries.

The second variety of method in multiplication is known as 'the Bank.'

In this game three or four children take part. First, there is the operator, who actually does the sum; then there is the banker, who is in charge of the bank material (Fig. 33); there is also the scribe, who notes down the calculations; and there may be

a fourth, who will check the exchanges and make sure no mistake occurs. Usually there is also an interested little group of spectators.

The banking material consists of sixty-three slips of paper or card, which the banker sets out as in Fig. 33. These slips,

1	10	100	1000
2	20	200	2000
3	30	300	3000
4	40	400	4000
5	50	500	5000
6	60	600	6000
7	70	700	7000
8	80	800	8000
9	90	900	9000

FIG. 34

or the figures on them, should be differently coloured, say green for units, blue for tens, red for hundreds, as in other such slips previously used by the children (pp. 13-14).

The material provided for the operator is similar, but should have some distinguishing difference. Fig. 34 shows the multiplicand, and Fig. 35 the multiplier. Dr Montessori says there should be four sets of the latter, of different colours, with cards indicating 1, 10, 100, 1000, to correspond. From this material the operator chooses his sum, say 275×36, and then puts the rest of the material away. The 200 card is at the bottom, the 70 card above that, and the 5 above that, giving 275.

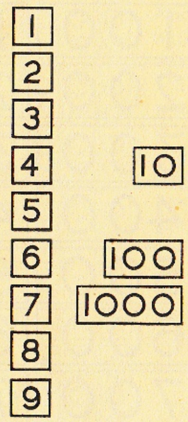


FIG. 35

Each quantity in the multiplicand has to be multiplied by each figure in the multiplier. The first step is to analyse the operation. The material is set out thus:

200	30
70	6
5	

Let us multiply first by the units:

200		
70	×	6
5		

The first product, 5×6, is 30; this is obtained from the banker and placed on a tray kept for the purpose. Similarly, the second product, $7 \times 10 \times 6$ (*i.e.*, 420) and the third, which is 1200, are received from the banker and placed on the tray.

We have now to multiply by 30. Multiplying first by 10, we may arrange our cards thus:

2000		
700	×	3
50		

We have now to receive from the banker 150, 2100, and 6000. When we ask the banker for 2100 he finds he can give us 2000, but not 100, because we already have the 100 slip. What is to be done? Looking among our slips we find we have 200. We give this back, and receive 300, which meets the difficulty. We now have to tidy up our results, as in our answer we may not have more than one figure in each hierarchy. We find we have no units; we have ten tens, which make 100; we give these back to the banker with our other 8 hundreds, and receive from him 900; we have 3 thousand slips, *viz.*, 1000, 2000, and 6000; in return for these slips we receive from the banker one 9000 slip. We now have only two cards on our tray, 9000 and 900. Superposing the 900 on the 9000, we get the answer, 9900.

It will be clear that the exchanges can be done at different times and in different ways. The whole activity is very pleasing to the children, and even adult students learning it for the first time can hardly be torn away from it.

FURTHER DISCOVERIES

Among the children in our ordinary schools there are lovers of arithmetic; there are also those who hate it, and there are many dejected travellers who accept what is given them with indifference if not with dislike. When the Montessori Method is used all become lovers; all accept the subject with lively interest and an enthusiasm for work.

In her *Advanced Method* Dr Montessori tells how when the material for the multiplication table was first presented to the children they became so enthralled that they demanded to be allowed to take it home with them. When this was not allowed and when they found it could not be purchased, one of the older girls proposed a strike. "The Dottoressa wants to try an experiment with us," she said. "Well, let's tell her that unless she gives us the material for the multiplication table we won't come to school any more."

In the ordinary school, especially in those in which class methods are still followed, the arithmetical pathway is beset by jungles and marshes, obscured by fog which hardly ever lifts to give a far-off glimpse of the goal. Many of the little travellers perish by the way, or struggle along overburdened and fatigued, supported or prodded by their teachers, but seeing no reward or purpose for all their pains.

By way of contrast, the Montessori pathway is on the heights; the clear air is exhilarating; the young travellers may linger here and there to examine their treasures more fully or go gaily forward to fresh discoveries.

The difficulties found in the teaching of arithmetic have led many teachers to try to associate it with the life of the children; one does not give them plain 'sums'; one introduces the number work by means of little stories which interest them and excite their imagination. 'Practical life' text-books are published. Difficult sums and long calculations are rejected, as being of little or no use in life.

But when this science of number is clearly and beautifully

presented, and when material is provided which enables the children freely to use their human power of reasoning, then it gives pleasure fully comparable to that felt in the development of any other power or skill.

Even in the earliest lessons we find foreshadowings of what is to come. We have already seen that from exercises with the divided rods we can pass directly to the summation of an infinite series. Here is another example of illumination due to the fact that we are working in the mountain air of pure science.

We take a number of the coloured bead-stairs, and on a black mat set out the multiplication tables as they are found in the Table of Pythagoras (p. 32). We have before us a beautiful, variegated, jewelled carpet. We notice that the bead-bars on the diagonal, in numbers 1, 4, 9, 16, etc., can be laid so as to form squares. For these, then, we substitute the corresponding bead-squares. Looking once more, we see that in the squares above and to the left of 4 we have a two-bead bar. If we place these on the top of the 4, we find we have a cube. Counting the beads in the squares above and to the left of 9 (3-square), we find we have 18, or twice 3-square. Again we can make a cube. With growing interest we proceed along the diagonal. In every case we find that the beads in the squares above and to the left of the diagonal will enable us to complete a cube on the diagonal. When we have finished our work all the beads, originally laid out as a carpet, are assembled on the diagonal in the form of cubes of gradually increasing size. We have discovered a general law.

The beads, as placed at first, are the square of the sum of the beads in the top row. Our rearrangement has shown that these same beads make up the sum of the cubes of the separate numbers. In other words, the square of the sum of the first powers of the first n integers is equal to the sum of their cubes —a most interesting, and, to the novice, a most surprising result, leading on to further queries and speculations.

TABLES OF MONEY, LENGTH, AREA, VOLUME, AND WEIGHT

Even at the nursery-school stage Dr Montessori teaches the children to do a little preliminary work in counting by means of the coins of the realm. She provides new money, but would have cardboard replicas if procurable. These objects are known in a general way to the child and are of great interest to him.

This is true in our country also. In an elementary school one can see the marked change in the children's expression when one gives a money sum that fits in with their experience. Children from poor homes have often what seems to some of us a precocious acquaintance with money; while it is possible for a boy of eight or nine, having a well-to-do home and attending a good school, to work money sums fluently, but to be unable to recognize the coins in general circulation.

In this country our tables of money, weights, and measures are highly individual and eccentric, as we have not adopted for daily use the scientific system. This conservatism of ours puts on our school children the extra work of learning the tables. This is a matter of little moment, as at their age they learn such things with ease, and are ready to be interested in the history embodied in our figures. What is more regrettable is that the scientific nomenclature is often not given them until the adolescent stage, when it is difficult for them to accustom themselves to it.

The schoolroom for young children should provide weights and measures so that children may by practice become familiar with them. In her reading exercises Dr Montessori has series of commands which might easily include some involving the act of measuring—*e.g.*, "Take two of your companions, measure them, and say which is the taller, and by how much." The actual numerical work involved in such calculations would be easy for children who have such facility and understanding of the decimal system as is given in the Montessori schools.

The decimal system, used for all weights and measures, including money, in Italy and other scientifically minded and progressive countries, is, in its concrete units of measurement, as well as in its theory, introduced to Montessori children at a very early age. In their nursery they have a pink tower of ten cubes which gradually decrease in size, from the largest, measuring ten centimetres along the side, to the smallest, the side of which measures one centimetre. The cubic centimetre, a universal unit for scientific measurement, is thus the baby block of the beloved pink tower. The largest block gives the volume of the litre. The longest rod of the series of rods (p. 10) is a metre; a thousand of them set end to end would make a kilometre. The affection felt for these objects, the deep-seated knowledge obtained of them by frequent handling and comparing, can hardly fail to affect favourably the attitude to the scientific measurements that are to come.

This careful preparation for the future, of which we have already given two examples (pp. 11, 61), is characteristic of the Method.

At every stage the Method is based on the study of the concrete. For the study of areas, square-root, cube-root, and other matters included in the curriculum of the secondary school, material is provided which brings about in the mind of the student not only understanding, but a mastery which leads to unforeseen developments and discoveries. And most certainly children prepared in the way we have described will offer firm foundations on which to build the structure required by secondary school and university.

BOOKS DEALING WITH ARITHMETIC
BY MARIA MONTESSORI, M.D.

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